Lesson 2 – Linear and Exponential Functions

Think about your own family. Everyone in your family has things about them that make them unique. But now think about what makes your family similar. Maybe it's the way you speak, or dress, or behave that makes the people in your family similar.

Just like you, certain functions belong to a family. In Algebra 1, we focus on four main families: linear, exponential, quadratic, and absolute value.

This lesson will aim to deepen your understanding of the characteristics of linear and exponential function "families." We will also learn a technique, called **regression**, which will allow us to write a function rule using a set of coordinates.

Linear Functions

The following four situations are examples of linear functions.

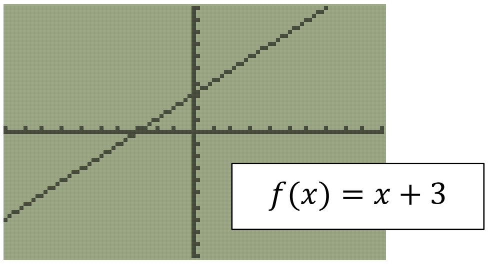
- Pedro earns $12 per hour at his job.
- A plant's height increases 3 inches per week.
- As water empties from a bathtub, its volume decreases at a rate of 2 gallons per minute.
- A car's distance traveled is changing at a constant speed of 55 miles per hour.

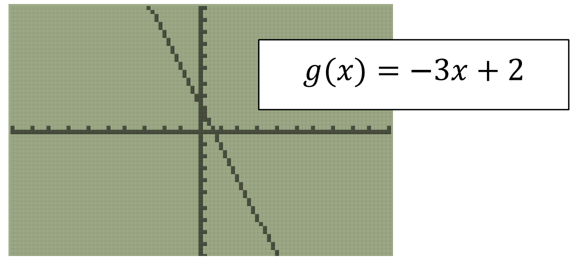
Write all the ways these situations are similar and how they are different.

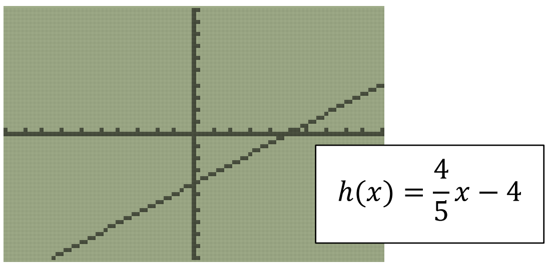
Each of these situations involve something that increases or decreases by a constant amount. Pedro's job, the plant's height, and the car's distance are all increasing, while the volume of water is decreasing. Each situation starts with a different amount and changes by a different amount.

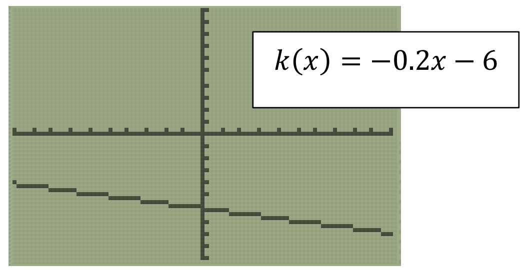
Educator Note: You may want to discuss the car's distance scenario in more detail as many students truly do not understand speed as a set amount of distance traveled each hour. Encourage students to represent each situation using a table and ask them probing questions about how a graph of each situation might look and why they think so.

The following graphs and rules are linear functions.









What is similar about the graphs of linear functions? What are some differences between the graphs?

Each graph is a perfectly straight line. Two of the lines slope (point) upward from left to right while the other two slope downward.

How are the linear function rules similar to one another? How are the linear function rules different?

Each rule has *x* in it. Each rule has a number that is added to or subtracted from the *x*-term. Three out of the four rules show a number in front of *x.* When there is no number in front of *x* that is showing it is 1, so all four rules have a number that multiplies *x* and a constant that is added or subtracted.

Educator Note: Lead a discussion to help students connect that the lines slope down when you multiply ***x*** by a negative and slope up when ***x*** is multiplied by a positive value. The number added or subtracted to the ***x***-term is the starting location on the ***y***-axis. However, if students don't notice it yet, don't feel the need to force the ideas.

Use your calculator the fill in tables for *f*, *g*, *h*, and *k* from $x = -5$ to $x = 5$.

x	$f(x) = x + 3$
-5	**-2**
-4	**-1**
-3	**0**
-2	**1**
-1	**2**
0	**3**
1	**4**
2	**5**
3	**6**
4	**7**
5	**8**

x	$g(x) = -3x + 2$
-5	**17**
-4	**14**
-3	**11**
-2	**8**
-1	**5**
0	**2**
1	**-1**
2	**-4**
3	**-7**
4	**-10**
5	**-13**

x	$h(x) = \frac{4}{5}x - 4$
-5	**-8**
-4	**-36/5**
-3	**-32/5**
-2	**-28/5**
-1	**-24/5**
0	**-4**
1	**-16/5**
2	**-12/5**
3	**-8/5**
4	**-4/5**
5	**0**

x	$f(x) = -0.2x - 6$
-5	**-5**
-4	**-5.2**
-3	**-5.4**
-2	**-5.6**
-1	**-5.8**
0	**-6**
1	**-6.2**
2	**-6.4**
3	**-6.6**
4	**-6.8**
5	**-7**

What similarities and differences do you notice about the tables of linear function rules?

When *x* increases by 1, the *y* changes by the number that multiplies the *x* in the rule. This is the rate of change.

The *y*-value when *x* is zero is the constant amount that is added to or subtracted from x. This is the starting value and the *y*-intercept.

Educator Note: Some students may state that $f(x)$ is increasing, so you may point out that when a number is becoming "more negative" it is actually decreasing. Temperature is a good analogy. If students struggle to reason with fractions, you may point out that $h(x) = \frac{4}{5}x - 4$ is equivalent to $y = 0.8x - 4$.

Every linear function rule can be written in the form

$$y = mx + b$$

The ***m*** and ***b*** represent numbers. Here are important ideas about them that you may have noticed from the similarities and differences work you just did.

m

- Also called the rate of change
- It is the slope of the graph
- How much the y-values increases or decreases by when the x-value increases by 1
- When m is positive, the graph increases as x increases from left to right
- When m is negative, the graph decreases as x increases from left to right

b

- The value of y when x = 0
- It is where the graph intersects the y-axis
- The starting amount for linear situations

Exponential Functions

Exponential functions are a type of nonlinear function, because their patterns do not result in straight lines. The following situations are examples of exponential functions.

A. 500 bacteria double every hour.
B. A car bought for $25000 is worth half of its value each year.
C. A savings account has $1000 initially and gains 5% interest annually.
D. A school with 900 students is decreasing its enrollment by 2% each year.

What do each of these situations have in common? How are they different?

Each situation is changing by multiplying or dividing. Situations A and C are increasing and situations B and D are decreasing. Situations C and D are changing by a percent while A and B multiply or divide by 2. B and C are situations involving money while A and D model population change.

Here are rules for each of the situations A through D.

$$A(x) = 500(2)^x$$

$$B(x) = 25000\left(\frac{1}{2}\right)^x$$

$$C(x) = 1000(1.05)^x$$

$$D(x) = 900(0.98)^x$$

How are these exponential rules similar? How are they different?

Each rule has an x as an exponent. Each rule shows a constant times another number that is raised to the power of x. Students may point out that one rule has a fraction in the parentheses, two rules have decimals in the parentheses, and one rule uses a whole number.

What connections do you see between the situation described and its function rule?

The starting amount is the number that appears outside of the parentheses. The amount that each situation multiplies by to get the next amount is the number that is in the parentheses.

Educator Note: The multiplier for situations A and B are more obvious than the multiplier for C and D. You may wish to explore situations C and D more with additional situations that lead students to see that when a situation changes by a percent, the multiplier is $1 + \frac{\%}{100}$ or $1 - \frac{\%}{100}$.

Here are tables for each of the rules.

$A(x) = 500(2)^x$

X	Y1
0	500
1	1000
2	2000
3	4000
4	8000
5	16000
6	32000

$B(x) = 25000\left(\frac{1}{2}\right)^x$

X	Y1
0	25000
1	12500
2	6250
3	3125
4	1562.5
5	781.25
6	390.63

$C(x) = 1000(1.05)^x$

X	Y1
0	1000
1	1050
2	1102.5
3	1157.6
4	1215.5
5	1276.3
6	1340.1

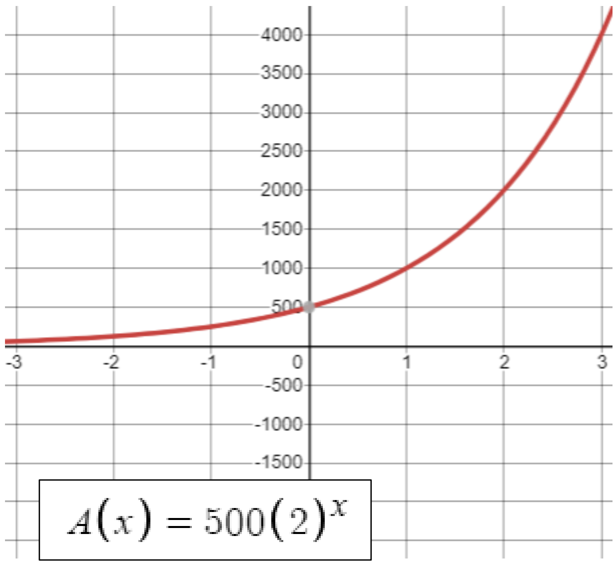
$D(x) = 900(0.98)^x$

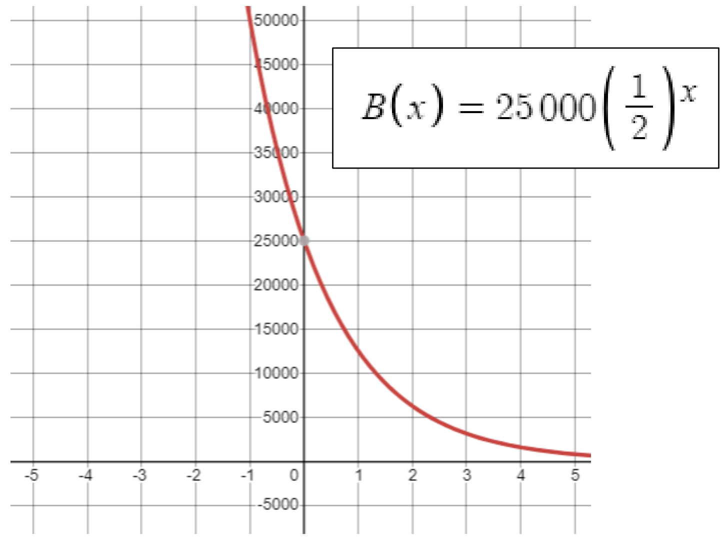
X	Y1
0	900
1	882
2	864.36
3	847.07
4	830.13
5	813.53
6	797.26

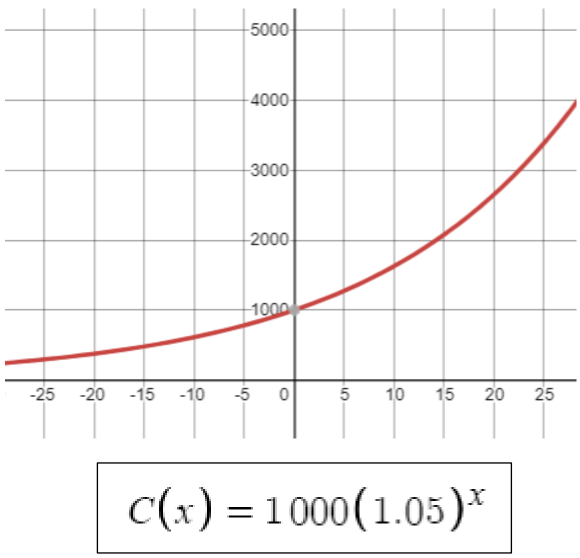
What similarities and differences do you notice between the exponential patterns in the tables? What connections do you notice about the function rule and the table of values?

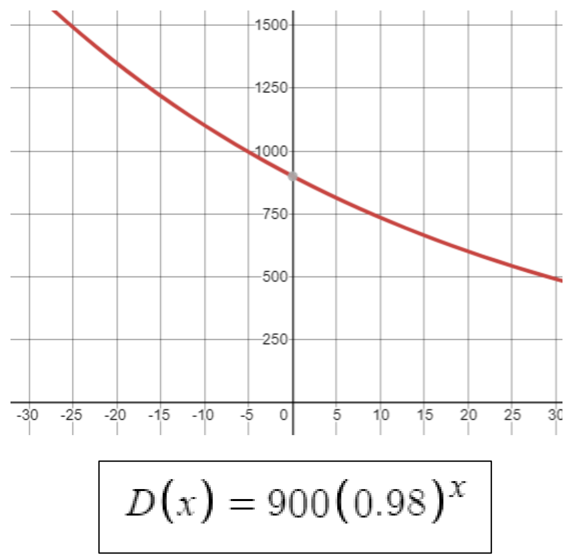
Each table starts when x is 0 at the number that matches the value outside of the parentheses (without the exponent) in the function rule. As x increases by 1, the y-value multiplies by the amount inside the parentheses.

Here are graphs of each exponential function.









What are some similarities and differences you notice about how each graph is shaped?

Students may describe graphs as being L-shaped or J-shaped, which is fine. Two of the graphs are decreasing from left to right while the other two are increasing.

How do the exponential graphs compare to the linear graphs?

Unlike linear graphs that increase or decrease at a constant amount, exponential graphs are not perfectly straight and increase or decrease by different amounts as x increases. You may want to lead a discussion about how exponential graphs that increase do so at a faster and faster rate. Exponential graphs that decrease do so slower and slower as x increases.

Similar to linear graphs, exponential graphs always increase or decrease. They never change direction.

What are some observations you can make about the how the exponential function rules and their graphs are related?

The number outside the parentheses is the y-intercept. The graph will increase if the number in the parentheses is more than 1 and it will decrease if the number in parentheses is between 0 and 1.

Every exponential function rule can be written in the form

$$\boldsymbol{y = a(b)^x}$$

Where **a** and **b** are numbers. Here are some important ideas about how these numbers affect the behavior of tables and graphs of exponential function rules.

a

- This is the initial (starting) value of the function
- On a table, this is the y-value when x = 0
- On a graph, this is where the graph will intersect the y-axis (vertical axis)

b

- This is the multiplier of the function
  If the function is growing or shrinking by a percent, b will be equal to $1 + \frac{\%}{100}$ or $1 - \frac{\%}{100}$
- To get the next y-value in the table, multiply the current y-value by b
- If b is greater than 1, the graph will be increasing from left to right
- If b is between 0 and 1, the graph will be decreasing from left to right

Decide whether the rule belongs to the linear function family, the exponential function family, or neither. Use evidence from the calculator or your knowledge of mathematics to justify your answer.

A. $y = 4 + 2x$

Linear

B. $y = 2(x + 1)^2 - 5$

Neither (Quadratic)

C. $f(x) = 3^x$

Exponential

D. $g(x) = |x + 1| - 3$

Neither (absolute value)

E. $k(x) = 0.5x - 6 + x$

Linear

F. $p(t) = 5000(0.87)^x$

Exponential

Decide which situations below are best modeled by <u>linear</u> functions, <u>exponential</u> functions, or <u>neither</u>. Explain your reasoning.

i. A bamboo plant grows at a constant rate of 3 inches per day.

Linear since the rate is constantly adding 3

ii. An amusement park allows 50 people to enter every 30 minutes

Linear since the rate constantly adds the same amount

iii. The value of a cell phone decreciates by 3.5% each year.

Exponential since the value decreases by a percent

iv. A baseball tournament eliminates half of the teams after each round.

Exponential since the number of teams divides by 2 each round (multiplies by 0.5).

v. A football is kicked into the air from an initial height of 4 feet. It reaches a maximum height of 82 feet before it returns to the ground.

Neither since this situation involves something that increases and decreases.

Linear Regression

This technique will transform a set of data into a line of best fit.

For example, in January 2019, the Regents exam gave data on the height of a certain breed of dog based upon its mass (weight).

First, make a prediction. What do you think is the relationship between the height of a dog and its weight? Would a linear function make sense to model this relationship?

Invite all discussion and discourse. Students may say in general as height increases so does the weight of a dog, while other students may disagree.

Here is the question:

34 The data given in the table below show some of the results of a study comparing the height of a certain breed of dog, based upon its mass.

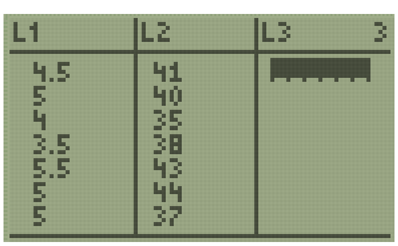
Mass (kg)	4.5	5	4	3.5	5.5	5	5	4	4	6	3.5	5.5
Height (cm)	41	40	35	38	43	44	37	39	42	44	31	30

Write the linear regression equation for these data, where x is the mass and y is the height. Round all values to the *nearest tenth*.

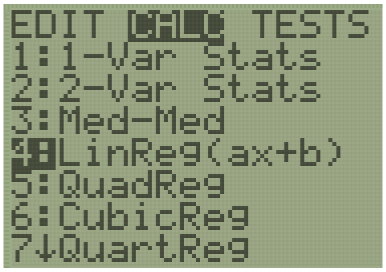
State the value of the correlation coefficient to the *nearest tenth*, and explain what it indicates.

The solution starts here. Follow along on your calculator as well.

Press **STAT**, **ENTER**



Type the x-values in **L1** and the y-values in **L2**

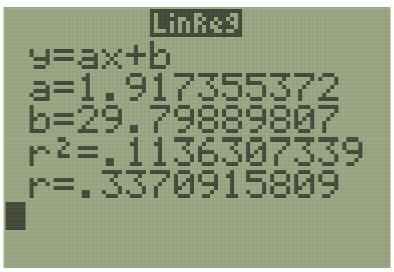


Press **STAT**, →

Since this problem asks for a linear regression, choose **4**

(Exponential regression is choice **0**)

Highlight “Calculate” and press **ENTER**



The first line, **y = ax + b**, is the function rule.

The next lines tell you the numbers to write in place of **a** and **b**. **x** and **y** will remain letters.

Ignore r^2

“r” is called the **correlation coefficient**, and it is always between 0 and 1. The closer **r** is to 1, the better fit this data is to the type of function rule you chose.

Notice that this question asked us to round the values of the function rule to the nearest tenth.

The linear regression equation is $\boldsymbol{y = 1.9x + 29.8}$

The correlation coefficient, to the nearest tenth, is **0.3**

Since 0.3 is not close to 1, this is a weak correlation.

Notice how the *r*-value 0.3 was closer to zero than one. This tells us that there is a weak relationship between the breed’s height and weight.

To better see this, do the following:

Step 1: When you press **Y=** be sure that **“Plot1”** is highlighted. If it isn’t, use the **arrow keys** to highlight it and press **ENTER**

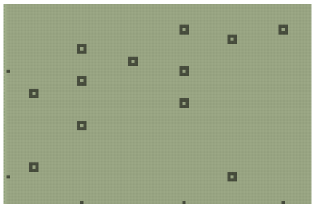
All **Y=** rules should be blank.

Step 2: Press **ZOOM**, and choose **"9: ZoomStat"** and then press **ENTER**

Describe the overall pattern of the data points. How do they compare to a linear function?

The points generally go up from left to right, but they definitely don't resemble a perfectly straight line.

Educator Note: Encourage students to identify which point might be the Chihuahua or the Great Dane and why they think that is.



Next, to see how well our function fits the data, type $y = 1.9x + 29.8$ into **Y1** by pressing **Y=**. This is the line that best fits our data. You can see that the line traces the pattern in the data, but that the points are relatively spread out. This makes for a weak correlation.

When working with real world measurements, or data, it is rare that the measurements make perfectly linear patterns. However, data is often close to linear, which is why the correlation coefficient is important. It tells us how well our data fits a line.

The correlation coefficient, **r**, is either strong or weak depending on whether it is closer to zero or one.

Complete the following problems on your own.

36 The percentage of students scoring 85 or better on a mathematics final exam and an English final exam during a recent school year for seven schools is shown in the table below.

Percentage of Students Scoring 85 or Better	
Mathematics, x	**English, y**
27	46
12	28
13	45
10	34
30	56
45	67
20	42

Write the linear regression equation for these data, rounding all values to the *nearest hundredth*.

$$y = 0.96x + 23.95$$

State the correlation coefficient of the linear regression equation, to the *nearest hundredth*. Explain the meaning of this value in the context of these data.

$$r = 0.92$$

There is a strong positive correlation between the percentage of students who score an 85 or higher on the math and English exams.

35 The table below shows the number of hours ten students spent studying for a test and their scores.

Hours Spent Studying (x)	0	1	2	4	4	4	6	6	7	8
Test Scores (y)	35	40	46	65	67	70	82	88	82	95

Write the linear regression equation for this data set. Round all values to the *nearest hundredth*.

$$y = 7.79x + 34.27$$

State the correlation coefficient of this line, to the *nearest hundredth*.

$$r = 0.98$$